Modified Euler is Consistent

Conts: $y' = f(t,y)$, $a < t < b$, $y(a) = \alpha$.

Discrete: $w_{i+1} = w_i + \frac{h}{2}\Big[f(t_i, w_i) +$

$+ f(t_{i+1}, w_i + h f(t_i, w_i))\Big]$

$w_0 = \alpha$.

Replacing w_i by y_i, and dividing by h we obtain the LTE.

$$\tau_{i+1}(h) = \left[\frac{y_{i+1} - y_i}{h} - \frac{1}{2}\left[f_i + f(t_i+h, y_i + h f_i)\right]\right]$$

$$- \left[y'(t_i) - f(t_i, y(t_i))\right]$$

Then,

$$\lim_{h\to 0} \tau_{i+1}(h) = y'(t_i) - \frac{1}{2}\left(f_i + f_i\right) - \left(y'(t_i) - f_i\right)$$

$$= 0. \checkmark$$

How do we obtain and explicit expression for $\tau_i(h)$.?

1.2 Definition: Convergence

The FDM (4) is convergent with respect to the IVP (3) if

$$\lim_{h\to 0} \max_{1\le i\le N} |w_i - y(t_i)| \to 0$$

1.3 Definition: Stability

The FDM (4) is stable if given two solutions $\{u_i\}_0^N$ and $\{v_i\}_0^N$ such that their initial values satisfy $|u_0 - v_0| < \epsilon$, then there exists $K > 0$ such that

$$|u_i - v_i| < K\epsilon, \qquad 1 \le i \le N$$

It means that small changes in the initial conditions produce correspondingly small changes in the subsequent approximations. Compare with well-posedness of an IVP.

1.4 Theorem 5.20: Stability-Convergence-Consistency

If there is $h_0 > 0$ such that $\phi(t, w, h)$ of FDM (4) is continuous and satisfies a Lipschitz condition in w with constant L on the set

$$D = \{(t, w, h) \mid t \in [a, b], w \in (-\infty, \infty), 0 \le h \le h_0\}$$

Then,

1. The method is stable.
2. The FDM is convergent if and only if it is consistent. This is equivalent to
$$\phi(t, y, 0) = f(t, y), \qquad a \le t \le b$$
3. If a function $\eta(h)$ exists such that $|\tau_i(h)| \le \eta(h)$ when $0 \le h \le h_0$, then
$$|y(t_i) - w_i| \le \frac{\eta(h)}{L} e^{L(t_i - a)}$$

1.5 Consistency and Convergence of Multistep Methods

Consider a general m-step multistep method

$$\begin{aligned} &w_0 = \alpha, \quad w_1 = \alpha_1, \ldots \quad w_{m-1} = \alpha_{m-1}, \\ &w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \ldots a_0 w_{i-(m-1)} + hF(t, h, w_{i+1}, w_i \ldots w_{i-(m-1)}), \end{aligned} \tag{5}$$

for $i = m-1, m, \ldots N-1$.

Thm 5.20 (Proof)

Conts. Equation: $y'(t) = f(t,y)$, $a<t<b$
$y(a) = \alpha$ (*)

Discrete equations:
$w_{i+1} = w_i + h\,\phi(t_i, w_i)$
$w_0 = \alpha$ (**)

If $\phi(t,w,h)$ satisfies a Lipschitz condition in w and is continuous in D, then

For $\{u_i\}_0^N$ and $\{v_i\}_0^N$ solutions of (**), such that $|u_0 - v_0| < \varepsilon$, $\varepsilon < \varepsilon_0$.

Then for any $h < h_0$

$$|u_i - v_i| = \left| u_{i-1} - v_{i-1} + h\left(\phi(t_{i-1}, u_{i-1}, h) - \phi(t_{i-1}, v_{i-1}, h)\right)\right| \le$$

$$\le |u_{i-1} - v_{i-1}| + hL\,|u_{i-1} - v_{i-1}| = (1+hL)\,|u_{i-1} - v_{i-1}|$$

$$\le (1+hL)^2 |u_{i-2} - v_{i-2}| \le \ldots \le (1+hL)^i |u_0 - v_0|$$

$$< (1+hL)^i \varepsilon \le (1+hL)^N \varepsilon = \left(1 + \frac{b-a}{N} L\right)^N \varepsilon$$

$$< \lim_{\substack{h\to 0\\ N\to\infty}} \left(1 + \frac{b-a}{N}\right)^N \varepsilon = e^{b-a} \varepsilon = K\varepsilon$$

if $K = e^{b-a}$

Modified Euler is Stable

$$w_{i+1} = w_i + \frac{h}{2}\Big[f(t_i,w_i) + f(t_{i+1}, w_i + hf(t_i,w_i))\Big] = w_i + h\,\phi(t_i,w_i,h)$$

We need to prove that ϕ satisfies a Lipschitz condition and ϕ is also conts in D.

<u>Lipschitz condition</u>: Assume $f(t,w)$ satisfies a Lipschitz cond. in w.

and Consider the pair (t,w) and $(t,\bar{w})$ in D

$$|\phi(t,w,h) - \phi(t,\bar{w},h)| = \frac{1}{2}\Big|f(t,w) - f(t,\bar{w}) + f(t+h, w+hf(t,w)) - f(t+h, \bar{w}+hf(t,\bar{w}))\Big| \le$$

$$\le \frac{1}{2}L|w-\bar{w}| + \frac{1}{2}L\big|w+hf(t,w) - (\bar{w}+hf(t,\bar{w}))\big|$$

$$\le \frac{1}{2}L|w-\bar{w}| + \frac{1}{2}L|w-\bar{w}| + \frac{h}{2}L|f(t,w)-f(t,\bar{w})|$$

$$\le L|w-\bar{w}| + \frac{h}{2}L^2|w-\bar{w}| = \left(L+\frac{h}{2}L^2\right)|w-\bar{w}|$$

$$= \hat{L}|w-\bar{w}|.$$

<u>Continuity of ϕ</u>: $\phi(t,w,h)$ is clearly conts. if $f(t,w)$ is continuous.

$\overset{\text{thm 5.20}}{\Longrightarrow}$ <u>Mod. Euler is Stable</u>

Multistep Methods. Stability and convergence

We start with a particular case:

$\boxed{\text{A-B 4th order}}$

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad w_3 = \alpha_3$$

$$w_{i+1} = w_i + \frac{h}{24}\Big[55 f(t_i, w_i) - 59 f(t_{i-1}, w_{i-1}) + 37 f(t_{i-2}, w_{i-2}) - 9 f(t_{i-3}, w_{i-3})\Big]$$

$$i = 3, 4, .., N-1 \qquad (M.1)$$

or

$$\boxed{w_{i+1} = w_i + h F(t_i, h, w_i, w_{i-1}, w_{i-2}, w_{i-3}) \qquad i = 3, 4, .., N-1. \qquad (M.2)}$$

Def. (Lipschitz condition)

We say that F satisfies a Lipschitz Cond. with respect to $\{w_j\}$ if there is $L > 0$ such that any pair of sequences $\{v_j\}_0^N$, $\{\tilde{v}_j\}_0^N$ and for $i = 3, 4, .., N-1$, we have

$$\boxed{\left|F(t_i, h, v_i, v_{i-1}, v_{i-2}, v_{i-3}) - F(t_i, h, \tilde{v}_i, \tilde{v}_{i-1}, .., \tilde{v}_{i-3})\right| \le L \sum_{j=0}^{3} |v_{i-j} - \tilde{v}_{i-j}| \qquad (M.3)}$$

Exercise #2

AB 4^{th} order

F satisfies a Lipschitz condition with respect to $\{w_j\}$ if $f(t,y)$ satisfies a Lipschitz cond. in y.

In fact,

$$\left|F(t_i, h, v_i, \ldots v_{i-3}) - F(t_i, h, \tilde{v}_i, \ldots \tilde{v}_{i-3})\right|$$

$$= \Big| \frac{55}{24}\big(f(t_i, v_i) - f(t_i, \tilde{v}_i)\big) - \frac{59}{24}\big(f(t_{i-1}, v_{i-1}) - f(t_i, \tilde{v}_{i-1})\big)$$

$$+ \frac{37}{24}\big(f(t_{i-2}, v_{i-2}) - f(t_{i-2}, \tilde{v}_{i-2})\big)$$

$$- 9\big(f(t_{i-3}, v_{i-3}) - f(t_{i-3}, \tilde{v}_{i-3})\big)\Big|$$

Using Lips. cond. in f

$$\le \frac{55}{24} L |v_i - \tilde{v}_i| + \frac{59}{24} |v_{i-1} - \tilde{v}_{i-1}|$$

$$+ \frac{37}{24} L |v_{i-2} - \tilde{v}_{i-2}| + \frac{9}{24} |v_{i-3} - \tilde{v}_{i-3}|$$

$$\le \underbrace{\frac{59}{24} L}_{L} \sum_{j=0}^{3} |v_{i-j} - \tilde{v}_{i-j}|$$

For a generalizations of the definition of the Lipschitz cond. for multistep methods read Burden's book.

Local Truncation error for A-B 4th order.

Replacing w_i by y_i in (M.1), we have after dividing by h.

$$\boxed{\frac{y_{i+1}-y_i}{h} - \frac{1}{24}\left[55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}\right] - \left[y'(t_i) - f(t_i, y(t_i))\right] = \tau_{i+1}(h) = L.T.E.} \quad (M.4)$$

Then, when $h \to 0$.. and assuming $f(t,y)$ is Continuous in its two variables. and $y(t)$ is also conts.

$$\tau_{i+1}(h) \to \cancel{y'(t_i)} - \frac{1}{24}\Big[\underbrace{55f_i - 59f_i + 37f_i - 9f_i}_{=24f_i}\Big]$$

$$- \cancel{y'(t_i)} - \cancel{f_i} = 0$$

Then, AB 4th order is Consistent with the ODE: $y' = f(t,y)$.

Consider a general multistep method

$$\boxed{w_0 = \alpha, \quad w_1 = \alpha_1, \quad .., \quad w_{m-1} = \alpha_{m-1}}$$

$$\boxed{\begin{aligned} w_{i+1} &= a_{m-1} w_i + a_{m-2} w_{i-1} \cdots + a_0 w_{i-(m-1)} \\ &\quad + h F(t_i, h, w_{i+1}, w_i, .., w_{i-(m-1)}), \\ &\qquad i = m-1, .., N-1. \end{aligned}} \qquad (M.5)$$

i) Local Truncation Error is defined as (replacing $w_i = y_i$, and dividing by h).

$$\boxed{\tau_{i+1}(h) = \left[\frac{y_{i+1} - a_{m-1} y_i - a_{m-2} y_{i-1} - \cdots - a_0 y_{i-(m-1)}}{h} - F(t_i, h, y_{i+1}, .., y_{i-(m-1)})\right] - \left[y'(t_i) - f(t_i, y_i)\right]} \qquad (M.6)$$

Def. (Consistency)

It consists of

i) Consistency of the one-step starting method.

and

ii) $\lim_{h \to 0} \tau_{i+1}(h) = 0$

$\boxed{\lim_{h\to 0} \tau_{i+1}(h) \to 0}$. It requires

a) $\boxed{\lim_{h\to 0} \dfrac{y_{i+1} - a_{m-1}y_i - a_{m-2}y_{i-1} - \dots - a_0 y_{i-(m-1)}}{h} = y'(t_i)}$

and

b) $\boxed{\lim_{h\to 0} F(t_i, h, y_{i+1}, y_i, \dots, y_{i-(m-1)}) = f(t_i, y_i)}$

This means that the finite difference (discrete formula) be an approximation to $y'(t)$ of certain order $\mathcal{O}(h^p)$.

Consistency and Convergence of A-B Methods: Explicit, Implicit and predictor-correctors.

Consider the Adams Predictor-Corrector method with m-steps.

Predictor: $w_{i+1} = w_i + h\left[b_{m-1}f_i + b_{m-2}f_{i-1} + \dots + b_0 f_{i-(m-1)}\right]$

corrector: $w_{i+1} = w_i + h\left[\tilde{b}_{m-1}f_{i+1} + \tilde{b}_{m-2}f_i + \dots + \tilde{b}_0 f_{i-(m-2)}\right]$

with $\tau_{i+1}(h) = \mathcal{O}(h^m)$ and $\tilde{\tau}_{i+1}(h) = \mathcal{O}(h^m)$ their L.T.E, respectively.

IVP: $y' = f(t,y)$, $y(a) = \alpha$, $a < t < b$.

Theorem 5.21

If $f(t,y)$ and $f_y(t,y)$ are conts. on

i) $D = \{(t,y) : a \le t \le b, \ -\infty < y < \infty\}$

ii) $|f_y| \le M$, on D.

Then, the L.T.E. of the P-C method is given by

a) $\sigma_{i+1}(h) = \tilde{\tau}_{i+1}(h) + \tau_{i+1}(h)\tilde{b}_{m-1} f_y(t_{i+1}, \theta_{i+1})$

and there exists constants K_1 and K_2 such that

b) $|w_i - y_i| \le \left[\max_{0 \le j \le m-1} |w_j - y_j| + K_1 \sigma(h) \right] e^{K_2(t_i - a)}$

It means the P-C method is consistent with $O(h^m)$.
and the P-C method is convergent, and the order of convergence is the lower between

Order of convergence of starting method

and $\sigma(h) = \max_{m \le i \le N} |\sigma_i(h)|$.

Stability of multistep methods. in general.

Relationship between stability, consistency and convergence

Consider the difference equation corresponding to the first part of (M.5)

$$\boxed{w_{i+1} = a_{m-1} w_i + a_{m-2} w_{i-1} + \dots + a_0 w_{i-(m-1)} \quad i = m-1, \dots, N-1} \qquad (M.7)$$

We look for solutions of (M.7) in the form of

$$\boxed{w_n = \lambda^n} \qquad (M.8)$$

Subst. into (M.7).

$$\lambda^{i+1} - a_{m-1}\lambda^{i} - a_{m-2}\lambda^{i-1} + \dots + a_0 \lambda^{i-(m-1)} = 0$$

or

$$\lambda^{i-(m-1)}\left[\lambda^{m} - a_{m-1}\lambda^{m-1} + \dots + a_0\right] = 0$$

So, for $w_n = \lambda^n$ to be a solution of (M.7) λ should be a root of the polynomial:

$$\boxed{P(\lambda) = \lambda^{m} + a_{m-1}\lambda^{m-1} + \dots + a_0.} \qquad (M.9)$$

If $\lambda_1, \lambda_2, \dots, \lambda_m$ are distinct roots of $P(\lambda)$, then every solution of (M.7) can be expressed as

$$\boxed{w_n = \sum_{i=1}^{m} c_i \lambda_i^{n}} \qquad (M.10)$$

1.11 Definition: Root Condition

If $\lambda_1, \dots \lambda_m$ are the roots of the characteristic polynomial

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \cdots - a_1\lambda - a_0$$

corresponding to the multistep method (5),

i) $|\lambda_i \leq 1 \qquad i = 1, \dots m$ and

ii) all roots with absolute value 1 are simple roots.

Then, the difference method is said to satisfy a **root condition**.
Discuss this concept.

1.12 Definition: Stability

1. Multistep methods that satisfy the root condition and have $\lambda = 1$ as the only root of the characteristic equation of magnitude 1 are called strongly stable.
2. Multistep methods that satisfy the root condition and have more than one distinct root with magnitude 1 are called weakly stable.
3. Multistep methods that do not satisfy the root condition are called unstable.

1.13 Theorem 5.24: Stability, Consistency, and Convergence of General Multistep Methods

If the m-step multistep method

$$\begin{aligned}
&w_0 = \alpha, \quad w_1 = \alpha_1, \dots \quad w_{m-1} = \alpha_{m-1}, \\
&w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \dots a_0 w_{i-(m-1)} + hF(t, h, w_{i+1}, w_i \dots w_{i-(m-1)}),
\end{aligned} \tag{10}$$

for $i = m-1, m, \dots N-1$, is consistent then,
it is stable if and only if it is convergent.
Discuss it.

Standard Multistep method

$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \ldots, w_{m-1} = \alpha_{m-1}$ $\quad \underline{m \text{ steps}}$

$$w_{i+1} = a_{m-1} w_i + a_{m-2} w_{i-1} + \cdots + a_0 w_{i-(m-1)}$$
$$+ h F(t_i, h, w_{i+1}, \ldots, w_{i-(m-1)})$$

Characteristic polynomial:

$$\boxed{P(\lambda) = \lambda^m - a_{m-1} \lambda^{m-1} - a_{m-2} \lambda^{m-2} - \cdots - a_0}$$

Examples 1. A-B 4th order. (4 steps)

$$w_{i+1} = w_i + \frac{h}{24}\left[55 f_i - 59 f_{i-1} + 37 f_{i-2} - 9 f_{i-3}\right]$$

$P(\lambda) = \lambda^4 - \lambda^3$

roots $\lambda^4 - \lambda^3 = 0 \Leftrightarrow \lambda^3(\lambda - 1) = 0$ $\quad \lambda_1 = 1$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$

Strongly stable.

Example 2. Milne's method

$$w_{i+1} = w_{i-3} + \frac{4h}{3}\left[2 f_i - f_{i-1} + 2 f_{i-2}\right] \quad (4 \text{ steps}).$$

roots : $P(\lambda) = \lambda^4 - 1 \lambda^{4-4} = \lambda^4 - 1 = 0 \Leftrightarrow \lambda^4 = 1 \Rightarrow \lambda_1 = 1, \lambda_3 = i, \lambda_2 = -1, \lambda_4 = -i$

all roots $|\lambda|$ weakly stable.

Standard 4-steps multistep method. $(m=4)$

$$w_{i+1} = a_3 w_i + a_2 w_{i-1} + a_1 w_{i-2} + a_0 w_{i-3}$$
$$+ h F(t_i, h, w_{i+1}, w_i, w_{i-1}, w_{i-2}, w_{i-3})$$

$$P(\lambda) = \lambda^4 - a_3 \lambda^3 - a_2 \lambda^2 - a_1 \lambda^1 - a_0$$

A-B 4th order: $a_3 = 1,\ a_2 = a_1 = a_0 = 0$

$$\boxed{P(\lambda) = \lambda^4 - \lambda^3.}$$

Milne's 4th order: $a_3 = a_2 = a_1 = 0,\ a_0 = 1$

$$\boxed{P(\lambda) = \lambda^4 + 1}$$